Physics 606 Exam 1 Solution

These are the main steps.
Your solution is likely to be more detailed.

1. Use $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)$, $p = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^\dagger - a)$

$$(\Delta x)^2 = \langle n | x^2 | n\rangle - (\langle n | x | n\rangle)^2 \qquad [(\Delta x)^2 \equiv \langle (x - \langle x\rangle)^2\rangle$$
$$(\Delta p)^2 = \langle n | p^2 | n\rangle - (\langle n | p | n\rangle)^2 \qquad = \langle x^2 - 2x\langle x\rangle + \langle x\rangle^2\rangle$$
$$\qquad = \langle x^2\rangle - \langle x\rangle^2]$$
$$a|n\rangle = \sqrt{n}\,|n-1\rangle \ , \quad a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Then $\boxed{\langle n | x | n\rangle =} \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\underbrace{\langle n | n+1\rangle}_{=0} + \sqrt{n}\,\underbrace{\langle n | n-1\rangle}_{=0}\right) = \boxed{0}$

and similarly $\boxed{\langle n | p | n\rangle = 0}$, so

(a) $\boxed{(\Delta x)^2 =} \langle x^2\rangle = \frac{\hbar}{2m\omega}\langle n | (a^{\dagger 2} + a^\dagger a + a a^\dagger + a^2) | n\rangle$

$$= \frac{\hbar}{2m\omega}\left(0 + \sqrt{n}\cdot\sqrt{n} + \sqrt{n+1}\cdot\sqrt{n+1}\right)\underbrace{\langle n|n\rangle}_{=1}$$
$$= \boxed{\frac{\hbar}{2m\omega}\,(2n+1)}$$

(b) Similarly, $\boxed{(\Delta p)^2 = \frac{m\hbar\omega}{2}\,(2n+1)}$.

Then $\boxed{\Delta p\, \Delta x} = \sqrt{\frac{m\hbar\omega}{2}}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{2n+1}\sqrt{2n+1}$
$$\boxed{= \hbar\left(n + \tfrac{1}{2}\right)} .$$

2. Fourier transform of $f(x)$:

$$f(k) = \int dx\, f(x)\, e^{-ikx}$$
$$\longrightarrow \quad f(\vec{k}) = \int d^3x\, f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$$

(a) so, for $\delta^{(3)}(\vec{r} - \vec{r}_0)$, $\boxed{f(\vec{k}) =} \int d^3x\, \delta^{(3)}(\vec{r} - \vec{r}_0)\, e^{-i\vec{k}\cdot\vec{r}}$
$$= \boxed{e^{-i\vec{k}\cdot\vec{r}_0}}$$

(b) With this convention,

$$f(x) = \int \frac{dk}{2\pi}\, f(k)\, e^{ikx}$$
$$f(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, f(k)\, e^{i\vec{k}\cdot\vec{r}}$$
$$\boxed{\delta(\vec{r} - \vec{r}_0) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r} - \vec{r}_0)}}$$

3. (a) $\boxed{[X_H(t), P_H(t)]} = [x(t), p(t)]\cos^2\omega t - [p(t), x(t)]\sin^2\omega t$

$= i\hbar(\cos^2\omega t + \sin^2\omega t) \boxed{= i\hbar}$

(b) $\dfrac{dX_H(t)}{dt} = \dfrac{1}{i\hbar}[X(t), H(t)] + \dfrac{\partial X_H(t)}{\partial t}$, $H(t) = \dfrac{p(t)^2}{2m} + \frac{1}{2}m\omega^2 x(t)^2$,

$$\frac{\partial X_H}{\partial t} \equiv e^{iHt/\hbar}\frac{\partial X(t)}{\partial t}e^{-iHt/\hbar}$$

$$= \frac{1}{i\hbar}\Big([x(t)\cos\omega t, \tfrac{1}{2m}p(t)^2] - \frac{1}{m\omega}[p(t)\sin\omega t, \tfrac{1}{2}m\omega^2 x(t)^2]\Big) + x(t)(-\omega\sin\omega t) - \frac{1}{m}p(t)\cos\omega t$$

Since $[A, BC] = [A,B]C + B[A,C]$,

$$[x(t), p(t)^2] = [x(t), p(t)]p(t) + p(t)[x(t), p(t)] = 2i\hbar p(t)$$

and $[p(t), x(t)^2] = [p(t), x(t)]x(t) + x(t)[p(t), x(t)] = -2i\hbar x(t)$

so $\boxed{\dfrac{dX_H(t)}{dt}} = \dfrac{1}{i\hbar}\Big(\dfrac{1}{2m}\cos\omega t \cdot 2i\hbar p(t) + \dfrac{1}{2}\omega\sin\omega t \cdot 2i\hbar x(t)\Big) - \dfrac{1}{m}p(t)\cos\omega t - \omega x(t)\sin\omega t$

$= \boxed{0}$

(c) $\boxed{\dfrac{dP_H(t)}{dt}} = \dfrac{1}{i\hbar}\Big([p(t)\cos\omega t, \tfrac{1}{2}m\omega^2 x(t)^2] + [m\omega x(t)\sin\omega t, \dfrac{p(t)^2}{2m}]\Big)$

$-\omega p(t)\sin\omega t + m\omega^2 x(t)\cos\omega t$

$= \dfrac{1}{i\hbar}\Big(\tfrac{1}{2}m\omega^2\cos\omega t(-2i\hbar x(t)) + \tfrac{1}{2}\omega\sin\omega t(2i\hbar p(t))\Big)$

$-\omega p(t)\sin\omega t + m\omega^2 x(t)\cos\omega t$

$= \boxed{0}$

4. (a) $\frac{d^2}{dx^2} e^{-\alpha x^2} = \frac{d}{dx}(-2\alpha x) e^{-\alpha x^2} = (-2\alpha + 4\alpha^2 x^2) e^{-\alpha x^2}$

$$\Rightarrow \boxed{E(\alpha) =} \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx \left[-\frac{\hbar^2}{2m}(-2\alpha + 4\alpha^2 x^2) + V(x)\right] e^{-2\alpha x^2}$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2} \left[\frac{\hbar^2}{2m}\left(2\alpha \underbrace{\int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2}}_{=\left(\frac{\pi}{2\alpha}\right)^{1/2}} - 4\alpha^2 \underbrace{\int_{-\infty}^{\infty} dx\, x^2 e^{-2\alpha x^2}}_{=\frac{1}{4\alpha}\left(\frac{\pi}{2\alpha}\right)^{1/2}}\right) + \int_{-\infty}^{\infty} dx\, V(x)\, e^{-2\alpha x^2}\right]$$

$$= \boxed{\frac{\hbar^2}{2m} \underbrace{(2\alpha - \alpha)}_{=\alpha} + \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, V(x)\, e^{-2\alpha x^2}}$$

(b) $0 = \frac{dE(\alpha)}{d\alpha}$

$$= \frac{\hbar^2}{2m} + \frac{1}{2}\alpha^{-\frac{1}{2}} \left(\frac{2}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, V(x)\, e^{-2\alpha x^2}$$
$$+ \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, V(x)\, e^{-2\alpha x^2} (-2x^2)$$

$$\Rightarrow \frac{\hbar^2}{2m} = \left(\frac{2\alpha}{\pi}\right)^{1/2} \cdot 2 \int_{-\infty}^{\infty} dx\, V(x)\, x^2 e^{-2\alpha x^2}$$
$$- \left(\frac{1}{2\pi\alpha}\right)^{1/2} \int_{-\infty}^{\infty} dx\, V(x)\, e^{-2\alpha x^2}$$

$$\Rightarrow \boxed{E(\alpha) =} \left(\frac{2\alpha}{\pi}\right)^{1/2} \cdot 2\alpha \int_{-\infty}^{\infty} dx\, V(x)\, x^2 e^{-2\alpha x^2}$$
$$- \frac{1}{2}\left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, V(x)\, e^{-2\alpha x^2}$$
$$+ \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, V(x)\, e^{-2\alpha x^2}$$

$$= \boxed{\left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, V(x) \left(2\alpha x^2 + \frac{1}{2}\right) e^{-2\alpha x^2}}$$

< 0 since $V(x) < 0$ in some region